Semi-classical strings in Sasaki-Einstein manifolds

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ABSTRACT: We find point-like and classical string solutions on the $AdS_5 \times X^5$, where X^5 are the 5-dimensional Sasaki-Einstein manifolds $Y^{p,q}$ and $L^{p,q,r}$. The number of acceptable solutions is limited drastically in order to satisfy the constraints on the parameters and coordinates of the manifolds. We find the energy of the above solutions as function of spins and we see that these relations depend on the parameters of the Sasaki-Einstein manifolds transcendentally. A discussion on BPS solutions is presented as well.

Contents

1 Introduction

The idea that large N gauge theories may have a string theory description was proposed a long time ago [1], but the first concrete proposal was given by Maldacena with the conjectured AdS/CFT correspondence [2–4]. The first and most studied model was the duality between the large N limit (where $\lambda = g_{YM}^2 N$ is kept fixed), maximally supersymmetric, $\mathcal{N} = 4$ Yang Mills theory in four dimensions with gauge group SU(N), and type IIB supergravity on $AdS_5 \times S^5$ with N units of RR 5-form flux through S^5. The duality is so useful as the radius of curvature R of the AdS_5 space, and the 't Hooft coupling λ, are related through $\lambda = R^4/l_{str}^4$, where l_{str} is the string length. When the curvature radius R is much bigger than the length of the string, the worldsheet theory becomes classical and can be studied. In this regime, λ approaches infinity and the dual gauge theory is strongly coupled and not under best control, which makes the calculations on the gravity side valuable and the correspondence useful. On the other hand, this fact makes it difficult

to realize the concrete connection between the string theory and the gauge theory beyond the supergravity approximation.

Work in this direction was made in [5], where certain gauge theory operators with large R-symmetry charge was proposed to be dual to all type IIB string states in a RR-charged pp-wave background [6], which is a Penrose limit of $AdS_5 \times S^5$ [7]. A generalization came in [8], where with the use of classical solitons and an appropriate quantization of the string theory, it has been shown that there are some semiclassical limits where the string/gauge duality can be reliable. For short strings, where one can approximate AdS_5 by a flat metric near the center, the leading closed string trajectory is reproduced. On the other hand for long strings, the strings feel the metric near the boundary of AdS, and reproduce the logarithmic behavior of the scaling of the wavefunction renormalization and hence the anomalous dimension for operators with "dimension minus spin" equal to two. Based on this work, a further generalization was made by considering multi-spin string states [9, 10]. At this time, many papers were published with the aim of finding new close string solutions, for example see [11] and references inside.

Another significant step was the identification of the one loop scalar dilatation operator with the Hamiltonian of integrable SO(6) spin chains [12]. This has brought impressive quantitative agreements between the energy of certain string solutions and the anomalous dimensions for very long operators [13]. Almost one year later, it was shown [14] that the spin chain in a certain subsector, in the limit of a large number of sites, can be described by a sigma model which agrees with the sigma model obtained from the rotating string in the appropriate limit.

All these results lead to further study with the aim to extend the above analysis to non-conformal theories or conformal theories with less supersymmetries. Here we concentrate on the latter and mention briefly that a famous effort in this direction was made possible with the use of the Leigh-Strassler [15] marginal deformation. Lunin and Maldacena, using a special case of this deformation on the original gauge/gravity duality, and taking advantage of the global $U(1) \times U(1)$ symmetry that the deformed theory and background should respect, found a new duality between $\mathcal{N} = 1$ super Yang-Mills theory and a deformed background [16]. This development gave new grounds for further testing and analyzing the AdS/CFT duality. Some characteristic examples including instantons, mesons, gravitons, Wilson loop calculations are in [17]. Equally important was the extension of the duality by Frolov [18] to non-supersymmetric theories. In some cases this made it possible to examine whether or not certain properties of field theories are related to supersymmetry [19].

Another famous gauge/gravity correspondence for theories with less supersymmetries, is with the use of a class of backgrounds with at least $\mathcal{N} = 1$ supersymmetry, which are type IIB and with the form $AdS_5 \times X^5$, where X^5 is a Sasaki-Einstein manifold and are dual to superconformal gauge theories called quivers.

A Sasakian manifold is a Riemannian manifold whose metric cone is Kähler. A Kähler manifold is a Hermitean manifold (M, g), say of dimension m, whose Kähler form Ω, defined by

$$\Omega_p(X, Y) = g_p(J_p X, Y) \qquad X, Y \in T_p M, \tag{1.1}$$

where J_p is the almost complex structure, it is closed. This form can be used to prove that a complex manifold is orientable, since the real $2m$ form $\Omega \wedge \ldots \wedge \Omega$ vanishes nowhere and it serves as a volume element. Additionally, if the manifold M is compact and admits a Ricci flat metric, then its first Chern class must vanish and the manifold is called Calabi-Yau. The Sasaki-Einstein manifolds, are manifolds whose metric cones are Ricci flat and Kähler.

The first non-trivial example in the AdS/CFT correspondence with the use of these manifolds was made in the case of the manifold $T^{1,1}$ [20]. Significant progress has been made in Sasaki-Einstein backgrounds and their dual field theories, almost a year before the Lunin-Maldacena correspondence was formulated, when it was found that for five-dimensional Sasaki-Einstein manifolds Y, there is an infinite family of inhomogeneous metrics on $Y^{p,q} \cong S^2 \times S^3$, which is characterized by relatively prime positive integers p, q with $0 < q < p$ [21–23]. In this case, there is an effective action of a torus $T^3 \cong \mathrm{U}(1)^3$ on the $C(Y^{p,q})$ which preserves the symplectic form on it and the metric, since it is an isometry. In [24] there is an extensive discussion on the geometric features of these manifolds and in a paper [25] that followed some days after, the superconformal quiver gauge theories dual to type IIB string theory on $AdS_5 \times Y^{p,q}$ was proposed.

The spaces $Y^{p,q}$ are of cohomogeneity one, but the correspondence can be generalized to spaces with cohomogeneity two, called $L^{p,q,r}$ spaces [26]. These are characterized by the relative positive coprime integers p, q and r with $0 < p \leq q$, $0 < r < p + q$ and with p, q to be coprime to $s = p + q - r$ and have isometry $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$. The metrics $Y^{p,q}$ are a special case of $L^{p,q,r}$ where $p + q = 2r$.

Moreover, like all theories with at least a $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry, the toric quiver gauge theories and their gravity dual theories admit β deformations [27], which also give space for further analysis. The gravity dual backgrounds can be found by performing a TsT (T-duality, 'shift' where the parameter of deformation enter, T-duality) transformation involving two of the angles that parametrize the $\mathrm{U}(1)$ directions [18], or by using the T-duality group [28]. An extended discussion for β-deformed Sasaki-Einstein dualities is presented in [29], where giant gravitons are also analyzed.

Therefore, it is obvious that there is an extensive attempt to extend and understand better the AdS/CFT correspondence using theories other than $\mathcal{N} = 4$ supersymmetric Yang-Mills. In this paper we investigate semi-classical string solutions in general $Y^{p,q}$ and $L^{p,q,r}$ manifolds. Work in this direction has been done for the very special case of $AdS_5 \times T^{1,1}$ examined in [30–32]. Moreover, a study for the case of BPS massless geodesics and their dual long BPS operators has been done in [34] for $Y^{p,q}$ manifolds and in [35] for $L^{p,q,r}$. Dual giant gravitons have been studied in [33] and recently giant magnons and spiky strings moving in a sector of $AdS_5 \times T^{1,1}$ have been examined in [36].

In this paper we mainly work on the gravity side and examine the motion of the string along some $\mathrm{U}(1)$ directions in Sasaki-Einstein spaces which is localized at $\rho = 0$ in the AdS space. We will see that in some cases it is difficult to find acceptable string solutions, due to the constraints imposed on the Sasaki-Einstein parameters. For the solutions we find, we present the energy-spin relation. The energy expressed in the momenta depends on the manifold considered i.e. on p, q, r, We also present an extensive discussion on point-like BPS string solutions. Notice that we do not examine the string dynamics in AdS_5 which are

identical to the maximally supersymmetric case, since all the equations can be decoupled.

2 The backgrounds

2.1 $Y^{p,q}$ metrics

The Sasaki-Einstein metrics $Y^{p,q}$ on $S^2 \times S^3$ can be presented in the following local form [22]:

$$ds^2 = \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\psi - \cos\theta d\phi)^2$$
$$+ w(y)\left[d\alpha + f(y)(d\psi - \cos\theta d\phi)\right]^2, \tag{2.1}$$

or more compactly

$$ds^2 = ds^2(B) + w(y)[d\alpha + A]^2, \tag{2.2}$$

where

$$w(y) = \frac{2(a - y^2)}{1 - cy},$$
$$q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2},$$
$$f(y) = \frac{ac - 2y + y^2 c}{6(a - y^2)}. \tag{2.3}$$

For $c = 0$ the metric takes the local form of the standard homogeneous metric on $T^{1,1}$. Generally we can scale the constant c to 1 by a diffeomorphism, and this is what we do in the rest of the paper.

To make the space B a smooth complete compact manifold we should fix the coordinates appropriately [22]. The parameter a is restricted to the range

$$0 < a < 1. \tag{2.4}$$

To make the base B_4 an axially squashed S^2 bundle over the round S^2 one can choose the ranges of the coordinates (θ, ϕ, y, ψ) to be $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $y_1 \leq y \leq y_2$ and $0 \leq \psi \leq 2\pi$. The parameter ψ is the azimuthal coordinate on the axially squashed S^2 fibre and the round sphere S^2 parametrized by (θ, ϕ). Also, by choosing the above range for a, the following conditions of y are satisfied: $y^2 < a$, $w(y) > 0$ and $q(y) \geq 0$. The equation $q(y) = 0$ is cubic and has three real roots, one negative and two positive. Naming the negative root y_{q-} and the smallest positive root y_{q+} we must choose the range of the coordinate y to be

$$y_{q-} \leq y \leq y_{q+}, \tag{2.5}$$

with the boundaries corresponding to the south and north poles of the axially squashed S^2 fibre. Also, it is necessary to have p/q rational in order to have a complete manifold. Note that

$$y_{q+} - y_{q-} = \frac{3q}{2p} \equiv \xi, \tag{2.6}$$

where ξ is defined for later use. Therefore, if the roots y_{q+}, y_{q-} are rational we speak for quasi-regular Sasaki-Einstein manifolds with the property that the volume of these manifolds having a rational relation to the volume of the S^5. However the rationality of p/q can be achieved even in cases that the two roots are irrational which gives irregular Sasaki-Einstein metrics.

Using the expressions (A.1) presented in the appendix A, we can express y_{q-} in terms of ξ

$$y_{q-} = \frac{1}{2}\left(1 - \xi - \sqrt{1 - \frac{\xi^2}{3}}\right), \tag{2.7}$$

where $0 < \xi < \sqrt{3}$. Since y_{q-} is the root of the cubic, a can be expressed in terms of ξ

$$a = \frac{1}{18}\left(9 - 3\sqrt{9 - 3\xi^2} + 4\xi^2\sqrt{9 - 3\xi^2}\right), \tag{2.8}$$

and in order to ensure that y_{q+} is the smallest positive root we constrain ξ to the range $0 < \xi < 3/2$. If we prefer, we can express a in terms of p, q using (2.6)

$$a = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2}, \tag{2.9}$$

then the period of α is given by $2\pi l$ where

$$l = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}. \tag{2.10}$$

2.2 $L^{p,q,r}$ metrics

The metric of this manifold is [26]

$$ds^2_{p,q,r} = (d\xi + \sigma)^2 + ds^2_{[4]}, \tag{2.11}$$

where

$$ds^2_{[4]} = \frac{\rho^2}{4\Delta(x)}dx^2 + \frac{\rho^2}{h(\theta)}d\theta^2 + \frac{\Delta(x)}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2 \tag{2.12}$$

$$+ \frac{h(\theta)\sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi\right)^2 \tag{2.13}$$

and

$$\sigma = \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi, \tag{2.14}$$

$$\Delta(x) = x(\alpha - x)(\beta - x) - \mu, \tag{2.15}$$

$$\rho^2 = h(\theta) - x, \tag{2.16}$$

$$h(\theta) = \alpha\cos^2\theta + \beta\sin^2\theta. \tag{2.17}$$

Here p, q and r are relative positive coprime integers and $0 < p \le q$, $0 < r < p + q$ and p, q are coprime to $s = p + q - r$. The metrics depends on two non-trivial parameters since

α, β, μ are constants, and we can set one of them equal to a non-zero number by rescaling the other two and x. The function $\Delta(x)$ plays a similar role to the function $f(y)$ in the $Y^{p,q}$ manifold, so x should be restricted between the two lowest roots of $\Delta(x) = 0$, namely x_1 and x_2, where

$$x_1 < x < x_2 \, . \tag{2.18}$$

Moreover, in order to have a smooth geometry in 5 dimensions the parameters should satisfy $\alpha, \beta \geq x_2$ where $x_2 \geq x_1 \geq 0$, which imply the already presented inequalities for p, q, r. The constants appearing in the metric are related to the roots of $\Delta(x)$ as follows:

$$\mu = x_1 x_2 x_3, \qquad \alpha + \beta = x_1 + x_2 + x_3, \qquad \alpha\beta = x_1 x_2 + x_1 x_3 + x_2 x_3, \tag{2.19}$$

where x_3 is the other root of $\Delta(x)$.

The metrics (2.1) of $Y^{p,q}$ can be derived as a special case of $L^{p,q,r}$ when $p + q = 2r$, which implies $\alpha = \beta$. The coordinate transformation is

$$\bar{\xi} = -2\alpha, \ \ \bar{\phi} = \frac{\psi + \phi}{2} + 3\alpha, \ \ \bar{\psi} = \frac{\psi - \phi}{2} + 3\alpha, \ \ \bar{\theta} = \frac{\theta}{2}, \ \ \bar{x} = \frac{(2y+1)\bar{\alpha}}{3} \, , \tag{2.20}$$

with μ related to a by

$$\bar{\mu} = \frac{4}{27}(1 - a)\bar{\alpha}^3 \, , \tag{2.21}$$

where we redefine the coordinates and the constants of $L^{p,q,r}$ using bars, in order to distinguish them from the ones of $Y^{p,q}$.

We also present the metric of AdS_5 in the Hopf coordinate system although we will be using the time element only

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho (d\beta_1^2 + \cos^2 \beta_1 d\beta_2^2 + \sin^2 \beta_1 d\beta_3^2) \, , \tag{2.22}$$

where the ranges of the angles are $0 \leq \beta_1 \leq \pi/2$, $0 \leq \beta_2, \beta_3 \leq 2\pi$.

3 String solutions in $Y^{p,q}$ background

3.1 Equations of motion and conserved quantities

In this section we present some spinning string solutions in the $Y^{p,q}$ manifold. We will fix the angle θ and hence are allowing the string to move on a circle of the round sphere S^2 parametrized by the coordinate ϕ. On the squashed sphere the string can move on its azimuthal coordinate ψ, and sit at a constant value y_0 between the north and south poles. This value will be chosen carefully by solving the equations of motion. Finally, the string can move on the principle S^1 bundle over B parametrized by α. Notice that each of the directions that the string is allowed to spin has a U(1) symmetry. As usual the global time is expressed through the world-sheet time as $t = \kappa\tau$, and the string is localized at the point

$\rho = 0$. The Polyakov action in the conformal gauge is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[-(-\dot{t}^2 + t'^2) + \frac{1-y}{6}(-\dot{\theta}^2 + \theta'^2) + \frac{1}{wq}(-\dot{y}^2 + y'^2) \right.$$

$$+ \left(\frac{1-y}{6}s_\theta^2 + \frac{q}{9}c_\theta^2 + wf^2c_\theta^2 \right)(-\dot{\phi}^2 + \phi'^2) + \left(\frac{q}{9} + wf^2 \right)(-\dot{\psi}^2 + \psi'^2) + w(-\dot{\alpha}^2 + \alpha'^2)$$

$$\left. -2c_\theta \left(\frac{q}{9} + wf^2 \right)(-\dot{\psi}\dot{\phi} + \psi'\phi') + 2wf(-\dot{\alpha}\dot{\psi} + \alpha'\psi') - 2wfc_\theta(-\dot{\alpha}\dot{\phi} + \alpha'\phi') \right].$$

For convenience we do not write explicitly the dependence of y in the functions f, w and q. The classical equations of motion for constant θ and y take the form

$$\frac{1-y}{6}(-\dot{\phi}^2 + \phi'^2)s_{2\theta} + \left(\frac{q}{9} + wf^2 \right)\left(s_{2\theta}(\dot{\phi}^2 - \phi'^2) + 2s_\theta(-\dot{\psi}\dot{\phi} + \psi'\phi') \right)$$

$$+2wfs_\theta(-\dot{\alpha}\dot{\phi} + \alpha'\phi') = 0, \quad (3.1)$$

$$\frac{s_\theta^2}{6}(\dot{\phi}^2 - \phi'^2) + \left(\frac{Q}{9} + A_1 \right)\left(c_\theta^2(-\dot{\phi}^2 + \phi'^2) - \dot{\psi}^2 + \psi'^2 - 2c_\theta(-\dot{\psi}\dot{\phi} + \psi'\phi') \right)$$

$$+W(-\dot{\alpha}^2 + \alpha'^2) + 2A_3(-\dot{\alpha}\dot{\psi} + \alpha'\psi' - c_\theta(-\dot{\alpha}\dot{\phi} + a'\phi')) = 0, \quad (3.2)$$

$$\partial_\beta \left[\gamma^{\beta\delta} \left(w\partial_\delta\alpha + wf(\partial_\delta\psi - c_\theta\partial_\delta\phi) \right) \right] = 0, \quad (3.3)$$

$$\partial_\beta \left[\gamma^{\beta\delta} \left(\frac{1-y}{6}s_\theta^2\partial_\delta\phi + \left(\frac{q}{9} + wf^2 \right)(c_\theta^2\partial_\delta\phi - c_\theta\partial_\delta\psi) - wfc_\theta\partial_\delta a \right) \right] = 0, \quad (3.4)$$

$$\partial_\beta \left[\gamma^{\beta\delta} \left(\left(\frac{q}{9} + wf^2 \right)(\partial_\delta\psi - c_\theta\partial_\delta\phi) + wf\partial_\delta\alpha \right) \right] = 0, \quad (3.5)$$

where we have used the conventions

$$A_1 \equiv \partial_y(wf^2), \quad A_2 \equiv \partial_y((wq)^{-1}), \quad A_3 \equiv \partial_y(wf), \quad Q \equiv \partial_y q, \quad W \equiv \partial_y w \quad (3.6)$$

and we have written the three last equations in a more compact form since for the ansatze we will choose they are satisfied trivially. In addition to the above equations, we get two more which come from the variation of the action with respect to the worldsheet metric. These are equivalent to the components of the energy-momentum tensor being set to zero. The Virasoro constraints read

$$\frac{1-y}{6}s_\theta^2\dot{\phi}\phi' + \left(\frac{q}{9} + wf^2 \right)\left(c_\theta^2\dot{\phi}\phi' + \dot{\psi}\psi' - c_\theta(\dot{\phi}\psi' + \dot{\psi}\phi') \right) + w\dot{\alpha}\alpha'$$

$$+wf\left(\dot{\alpha}\psi' + \dot{\psi}\alpha' - (\dot{\alpha}\phi' + \dot{\phi}\alpha')c_\theta \right) = 0, \quad (3.7)$$

$$-\kappa^2 + \frac{1-y}{6}s_\theta^2(\dot{\phi}^2 + \phi'^2) + w(\dot{\alpha}^2 + \alpha'^2) + 2wf\left(\dot{\alpha}\dot{\psi} + \alpha'\psi' - (\dot{\alpha}\dot{\phi} + \alpha'\phi')c_\theta \right)$$

$$+ \left(\frac{q}{9} + wf^2 \right)\left(c_\theta^2(\dot{\phi}^2 + \phi'^2) + \dot{\psi}^2 + \psi'^2 - 2c_\theta(\dot{\phi}\dot{\psi} + \phi'\psi') \right)$$

$$+\frac{1-y}{6}(\dot{\theta}^2 + \theta'^2) + \frac{1}{wq}(\dot{y}^2 + y'^2) = 0, \quad (3.8)$$

where only in the last equation (3.8) we include the terms corresponding to a non-constant θ and y for later use in section 3.2.

The symmetry of $Y^{p,q}$ admits three conserved charges which are the angular momenta corresponding to strings rotating along the α, ϕ and ψ directions. Moreover, there exists one more conserved quantity, the classical energy, which is generated by the translational invariance along t. All of them are presented below:

$$E = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \, \kappa \,, \tag{3.9}$$

$$J_a = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \, (w\dot{a} - wfc_\theta\dot{\phi} + wf\dot{\psi})\,, \tag{3.10}$$

$$J_\phi = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \left(-wfc_\theta\dot{a} + \left(\frac{1-y}{6}s_\theta^2 + \frac{q}{9}c_\theta^2 + wf^2c_\theta^2 \right) \dot{\phi} - \left(\frac{q}{9} + wf^2 \right) c_\theta \dot{\psi} \right), \tag{3.11}$$

$$J_\psi = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \left(wf\dot{a} - \left(\frac{q}{9} + wf^2 \right) c_\theta \dot{\phi} + \left(\frac{q}{9} + wf^2 \right) \dot{\psi} \right) \,. \tag{3.12}$$

Let us also define the new quantities $J_{\text{tot}} = J_\alpha + J_\phi + J_\psi$, $\mathcal{E} = E/\sqrt{\lambda}$, $\mathcal{J}_i = J_i/\sqrt{\lambda}$ and $\mathcal{J}_{\text{tot}} = J_{\text{tot}}/\sqrt{\lambda}$ for later use. The conserved quantity corresponding to the total SU(2) angular momentum is

$$J^2 = J_\theta^2 + \frac{1}{s_\theta^2}(J_\phi + c_\theta J_\psi)^2 + J_\psi^2 \,. \tag{3.13}$$

Finally, for point-like stings localized at constant points on θ and y, the following identity holds

$$\sqrt{\lambda}\,\kappa^2 = \dot{\alpha} J_\alpha + \dot{\phi} J_\phi + \dot{\psi} J_\psi \,. \tag{3.14}$$

In the following we will choose an ansatz where the string is moving in $Y^{p,q}$ along the three angles α, ϕ, ψ and is at rest along all the other directions

$$\alpha = \omega_1\tau + m_1\sigma, \qquad \phi = \omega_2\tau + m_2\sigma, \qquad \psi = \omega_3\tau + m_3\sigma\,, \tag{3.15}$$

$$\theta = \theta_0 \ \text{ and } \ y = y_0, \tag{3.16}$$

where θ_0, y_0 are constants and their exact values should be chosen to be consistent with the solutions of the equations of motion and the Virasoro constraints. Notice also, that due to the periodicity condition in the global coordinates of the manifold on σ, the winding numbers have to be integers. For the linear dependence on τ, σ of (3.15), the equations of motion (3.3), (3.4) and (3.5) for α, ϕ and ψ respectively, are trivially satisfied.

3.2 Discussion on BPS solutions

In this section we discuss the BPS point-like solutions. The R-symmetry in the field theory is dual to the canonically defined Reeb Killing vector field K on the Sasaki-Einstein manifolds [24, 25], given by

$$K = 3\frac{\partial}{\partial\psi} - \frac{1}{2}\frac{\partial}{\partial\alpha} \tag{3.17}$$

and the R-charge is equal to

$$Q_R = 2J_\psi - \frac{1}{3}J_\alpha \,. \tag{3.18}$$

Now in order to express the Hamiltonian in terms of the momenta, we are initially considering a general situation, where all the parameters in the internal manifold are dependent on τ:

$$\theta = \theta(\tau), \qquad y = y(\tau), \qquad \alpha = \alpha(\tau), \qquad \phi = \phi(\tau), \qquad \psi = \psi(\tau), \qquad (3.19)$$

and later we will focus on the string configurations mentioned in the previous section. The reason we are doing this, is to show how the general BPS solutions behave if we generalize our ansatz and activate simultaneously the motion on all angles. The process is equivalent to finding massless geodesics and is examined in [33, 34].

We start by expressing the energy in terms of the momenta. Now since we consider motion on the θ and y coordinates, we have the-non zero conjugate momenta

$$J_y = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \, \frac{1}{w \, q} \dot{y}, \qquad J_\theta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \, \frac{1-y}{6} \dot{\theta} \ . \qquad (3.20)$$

It is straightforward to substitute in the second Virasoro constraint the velocities in terms of their momenta and get

$$\kappa^2 = \frac{1}{\lambda} \left(w \, q \, J_y^2 + \frac{6}{1-y}(J^2 - J_\psi^2) + \frac{1}{w}J_\alpha^2 + \frac{9}{q}(J_\psi - f J_\alpha)^2 \right) \ . \qquad (3.21)$$

The energy of the string is given by (3.9), and is equal to the conformal dimension Δ of the dual operator, and to find the lower bound of it, we should express (3.21) in terms of the R-charge. Using (3.18), we obtain via the algebra in (3.21):

$$\Delta^2 = \left(\frac{3}{2}Q_R \right)^2 + \frac{1}{wq}(J_a + 3yQ_R)^2 + wqJ_y^2 + \frac{6}{1-y}(J^2 - J_\psi^2) \ . \qquad (3.22)$$

Since $y_{q+} < 1$, which is the upper bound of y, and $J^2 \geq J_\psi^2$, all the terms in the above equation are positive, which leads to the inequality $\Delta \geq 3/2 \, Q_R$. The solutions generated by the equality correspond to BPS operators, and in order to saturate the bound, all the following equations must be satisfied

$$J_y = 0, \qquad J_\theta = 0, \qquad J_\phi = -c_\theta J_\psi, \qquad J_\alpha = -3yQ_R. \qquad (3.23)$$

The two first equations fix θ and y to unknown constants. The next two can be used to determine the relationship between α, ϕ, ψ using the constants y, θ. The situation now is getting closer to our initial configuration in the previous section where we considered y, θ as constant, with the difference that the τ dependence of the U(1) angles is now unknown, and needs to be determined by the equations of motion and the Virasoro constraints. In order to find BPS solutions we need to solve the equations of motion (24-28) together with (3.23) and use (3.8) to calculate the energy. The first Virasoro constraint (3.7) is trivially satisfied.

It is more convenient to proceed by solving first the equations (3.3), (3.4) and (3.5) for constant y and θ, since their solutions are very special. As we said above, these equations are satisfied trivially for angles with linear dependence on τ, i.e. the one written in (3.15)

where for the point-like case is equivalent to set all the m_i zero. The rest of the solutions we get constrain y to live on its maximum or minimum values, which means on the poles of the squashed sphere. More specifically the solutions are

$$\alpha = \omega_1 \tau, \qquad \phi = \omega_2 \tau, \qquad \psi = \omega_3 \tau \quad \text{or} \tag{3.24}$$

$$y = y_{q\pm}, \qquad \phi = \omega_2 \tau, \qquad \ddot{\alpha} = \frac{1-y}{6y} \ddot{\psi}. \tag{3.25}$$

However, when we take account of the boundary conditions in (3.25), we see that $\dot{\psi} = 0$, since we are at the poles of the squashed sphere. This fixes ψ to be constant and the other two U(1) angles to have linear dependence on τ. This makes (3.25) a special case of (3.24). So the only way to satisfy the above equations for the non-constant angles is to have a linear dependence on τ, where the θ and y are not yet fixed to a specific value.

Bearing the above results in mind, we proceed by finding general solutions that satisfy the two first equations of motion (3.1), (3.2), together with the BPS equations (3.23) and present some solutions, starting with a solution which was also found in [34]

$$\dot{\phi} = 0, \qquad \dot{\alpha} = -\frac{\dot{\psi}}{6}. \tag{3.26}$$

We see that this solution is valid for any y that satisfies the inequality (2.5), since in this case, the last equation of (3.23) is satisfied trivially. Additionally, the third equation in (3.23) is satisfied also trivially and hence θ can take any constant value inside the region where it is defined. However notice that in order for the solution to satisfy (26-28), the α and ψ angles must have a linear dependence on τ, and hence

$$\alpha = -\frac{\omega_3}{6} \tau, \qquad \psi = \omega_3 \tau. \tag{3.27}$$

The energy for this solution is equal to $E = \sqrt{\lambda} |\dot{\psi}|/3$ and the conserved momenta in this case are

$$J_\alpha = -\frac{2}{3} \sqrt{\lambda} y \dot{\psi}, \qquad J_\phi = \frac{1}{9} \sqrt{\lambda} (y-1) c_\theta \dot{\psi}, \qquad J_\psi = -\frac{1}{9} \sqrt{\lambda} (y-1) \dot{\psi} \tag{3.28}$$

and are related each other by

$$J_\phi = -c_\theta J_\psi = \frac{1-y}{6y} c_\theta J_\alpha . \tag{3.29}$$

In this case the energy can be written as

$$E = \frac{3}{|(y-1)(c_\theta - 1) - 6y|} |J_{\text{tot}}| . \tag{3.30}$$

Notice that in the above relation the factor of proportionality is independent of a, and hence on the manifold considered.

Another set of possible solutions different than (3.27) are:

$$\theta = 0, \qquad y = 1 \pm \frac{\sqrt{1-a}}{\sqrt{3}}, \qquad \dot{\alpha} = \frac{\dot{\phi} - \dot{\psi}}{6}, \tag{3.31}$$

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where again α, ϕ, ψ have to be linear with τ in order for the above expressions to satisfy (3.3), (3.4) and (3.5). The y solution is a special case of (3.33) for $\theta = 0$ and as we show in the next section, it is unfortunately always greater than y_{q+} except in the limit $y = y_{q+} = 1$. This is the case where $a = 1$ and we are not going to examine it any further. However, to be more accurate here, we have to set $\dot{\phi} = 0$, since for $\theta = 0$ we are on the pole of S^2 and there is no meaning in defining rotation along ϕ direction. Hence, the corresponding equation in (3.31) should be modified to $\dot{\alpha} = -\dot{\psi}/6$.

More interesting are the following solutions where y lives on its boundaries

$$y = y_{q\pm}, \qquad \theta = 0, \qquad \dot{\phi} = 0, \qquad \dot{\alpha} = -\frac{y+2}{6y}\dot{\psi}, \tag{3.32}$$

and the dependencies of the non-constant angles on τ are linear. This solution is acceptable since the inequality (2.5) is satisfied. However, by considering the boundary conditions, the solution become trivial, since we are on the poles of the squashed sphere where rotation along the ψ direction cannot be defined. Hence, we have to impose $\dot{\psi} = 0$ and then the whole string ansatz becomes static.

One could possibly find, other non-interesting solutions at the limits $a = 0, 1$, but it seems that there are no other BPS solutions than the ones presented above, which allow a to be at other points except zero and one. The solution (3.32) has as its main property to restrict y to the boundaries of (2.5) and to localise the string on the two three-submanifolds obtained by the initial manifold for $y = y_{q\pm}$, and denoted by $\Sigma_- \cong S^3/\mathcal{Z}_{p+q}, \Sigma_+ \cong S^3/\mathcal{Z}_{p+q}$. The cones over these Lens spaces are divisors of $C(Y^{p,q})$ and hence supersymmetric submanifolds [24]. This is because the induced volume form on Σ is equal to the four-form $\mathcal{J} \equiv J \wedge J/2$, where J here is the Kähler symplectic form. Hence, these cones are calibrated with respect to $\mathcal{J}$.

In the following sections, non-BPS point-like, as well as extended string solutions, will be examined.

3.3 One angle solution

In this section we examine the simplest case, where only the angle which parametrizes some U(1) direction of the manifold in this coordinate system is turned on. In this case it is known that classical spinning string solutions that wrap around the circle do not exist due to diffeomorphism invariance, or equivalently because the first Virasoro constraint forces the metric element in the spinning direction to vanish,[1] or make the string ansatz trivial. In $Y^{p,q}$ manifolds however, we just mention, that the diagonal metric elements in these three Killing vector directions can not vanish for y in the range (2.5) and hence the Virasoro constraints force the ansatz in that case to become static. Let us show briefly that the diagonal metric elements in the U(1) directions can not vanish.

For the $g_{\alpha\alpha}$, this is obvious since it is equal to zero for $y = \pm\sqrt{a}$ and we already know that these values do not satisfy (2.5). For the $g_{\phi\phi}$ element, the situation is more

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[1]Which conceptually does not look right since we consider motion along this direction only.
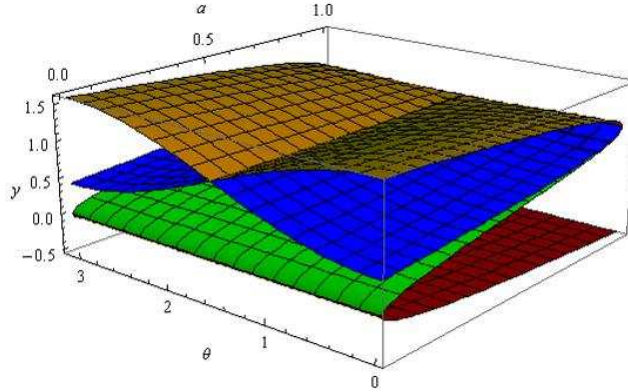
Figure 1. We plot $y_\pm = 1 \pm \sqrt{1-a}\,c_\theta^2/\sqrt{3}$ together with the y_{q_+}, y_{q_-} versus a, θ. The y_{q_-}, y_{q_+} surfaces are colored red and green respectively, while y_-, y_+ colored blue and orange respectively. From the figure we notice that both $y_\pm$ are greater than y_{q+} for the whole range of a and θ.

complicated since it is equal to zero for

$$y_\pm = 1 \pm \frac{\sqrt{1-a}\,c_\theta^2}{\sqrt{3}} \ . \tag{3.33}$$

Obviously the y_+ solution is discarded since is bigger than one, but also the y_- solution is outside the desirable area as can be seen in figure 1. The last diagonal metric element $g_{\psi\psi}$ is zero for $y_\pm = 1 \pm \sqrt{1-a}/\sqrt{3}$, where y_- is the lower bound of the previous solution and obviously this metric element cannot be zero too.

Let us now begin by looking for point-like string solutions. By allowing the string to move only along the α direction, and using the ansatz $\alpha = \omega_1 t$, the only non-trivially satisfied equation that we need to solve is (3.2), which takes the simple form:

$$\left(2 - \frac{2(a-1)}{(y-1)^2}\right)\omega_1^2 = 0\,, \tag{3.34}$$

and has solutions

$$y_\pm = 1 \pm \sqrt{1-a} \ . \tag{3.35}$$

These solutions are plotted with the $y_{q\pm}$ (figure 2), and it is obvious that only y_- is an acceptable solution, since $y_{q-} \le y_- \le y_{q+}$ for the whole range of a. Then, for $y = y_-$ the second Virasoro constraint (3.8) gives

$$\kappa^2 = w\,\omega_1^2 \Rightarrow \kappa^2 = 4\left(1 - \sqrt{1-a}\right)\omega_1^2 \ . \tag{3.36}$$

Then using (3.9) - (3.12), we get for energy and the non zero conserved charges

$$E = \sqrt{\lambda}\,\sqrt{w\,\omega_1^2}\,, \tag{3.37}$$

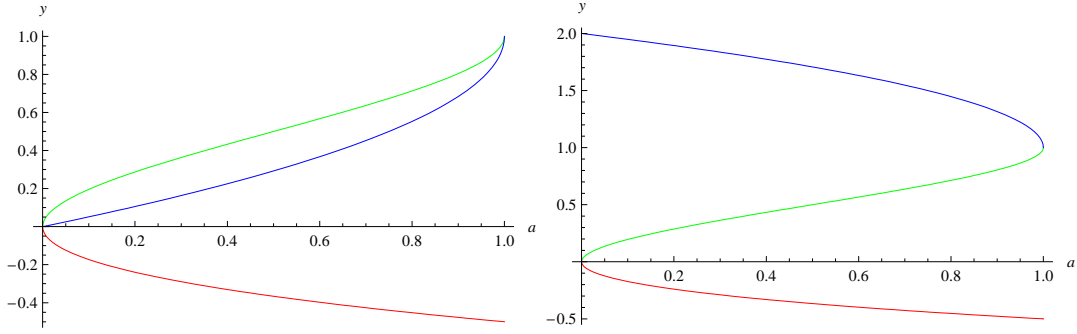$$J_\alpha = \sqrt{\lambda}\,w\,\omega_1 \ . \tag{3.38}$$

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Figure 2. Plotting the roots y (blue) of (3.35) with y_{q-} (red), y_{q+} (green) versus a. This color mapping to the solution will be the same in the other plots too. In the first plot we see that the smaller solution y_- is between the two roots of $q(y) = 0$ and in the second one that the greater root y_+ is outside the allowed area since $y_+ > 1$.

Combining the above relations we end up with

$$E = \sqrt{\frac{1}{w}} J_\alpha^2 \Rightarrow \quad E = \frac{\sqrt{\lambda}}{2} \sqrt{\frac{1}{1 - \sqrt{1-a}} \mathcal{J}_\alpha^2} \,. \tag{3.39}$$

Note that the energy depends linearly on $\mathcal{J}_\alpha = \mathcal{J}_{\text{tot}}$ and depends on the manifold $Y^{p,q}$, since a is related to p, q by the equation (2.8), with the factor of proportionality being a monotonically decreasing function with respect to a. On the other hand, the function of the energy in terms of a, ω_1 is a monotonically increasing function with respect to a as can be seen from (3.36). As a final remark, we mention that all the calculations are independent of the angle θ, which can be chosen as any constant angle.

For completeness we now can consider a static string along the α direction with $\alpha = m_1\sigma$. The solution of the equation of motion remains the same, given by (3.35). The only difference is that all the momenta are now equal to zero and the energy is expressed as

$$\kappa^2 = wm_1^2 \Rightarrow E = 2\sqrt{\lambda}\sqrt{\left(1 - \sqrt{1-a}\right) m_1^2} \,. \tag{3.40}$$

Following a similar procedure we choose a different angle ϕ with a similar ansatz and consider a point-like string motion with $\phi = \omega_2 t$. Then, the non-trivial equations are (3.1) and (3.2), which take the form

$$\frac{(a-1)s_{2\theta}}{18(y-1)}\omega_2^2 = 0 \,, \tag{3.41}$$

$$\frac{a - 7 - 6(y-2)y + (a-1)c_{2\theta}}{36(y-1)^2}\omega_2^2 = 0 \,, \tag{3.42}$$

and do not give any new real solution, since the first equation is solved for $\theta = n\pi/2$, $n = (0,1,2)$ and the second one can not be solved for real y. The situation is similar when we choose the static string ansatz $\phi = m_2\sigma$, where the above equations remain the same with ω_2 replaced by m_2, and obviously again they do not give any new solutions.

Finally, for the third angle which parametrizes the remaining U(1) direction, again consider the string ansatz $\psi = \omega_3 t$. The only non-trivial equation of motion is (3.2)

$$\frac{a - 4 - 3(y-2)y}{18(y-1)^2}\omega_3^2 = 0\,, \tag{3.43}$$

which has solutions $y = 1 \pm \sqrt{-1+a}/\sqrt{3}$ and are not real. The situation is similar for the ansatz $\psi = m_3\sigma$, since the equation (3.43) remains the same, with only difference the replacement of ω_3 by m_3.

Summarizing, if we restrict the string to rotate or wrap only along one U(1) direction in $Y^{p,q}$ as we have done above, there is only one possible configuration. It is the point-like string moving along the fibre direction $\alpha = \omega_1\tau$. This is one spin solution, where the energy is proportional to $\mathcal{J}_\alpha$ and depends on the manifold $Y^{p,q}$ in a way that the factor of proportionality is a decreasing function of a.

We expect that the solution (3.35) is not BPS. More specifically the Hamiltonian here is

$$H = \sqrt{\lambda}\frac{1}{2}w\omega_1^2 = 2\sqrt{\lambda}\left(1 - \sqrt{1-a}\right)\omega_1^2\,. \tag{3.44}$$

Hence the R-charge and the conformal dimension of the dual operator can be written as

$$Q_R = -\frac{\sqrt{\lambda}}{3}w\omega_1,\quad \Delta^2 = \left(\frac{3}{2}Q_R\right)^2 + \lambda(4a - w)\omega_1^2 = \left(\frac{3}{2}Q_R\right)^2 + \frac{\sqrt{1-a}}{4(1-\sqrt{1-a})}J_\alpha^2\,.$$

It is obvious that $\Delta > 3/2\,Q_R$ always, since the expression $(4a - w)$ is always positive, and become equal to zero only in the limits $a = 0,\ 1$. Hence, the solution is non-BPS.

3.4 The two angle solutions

Here we examine some solutions where the strings are moving along two U(1) directions. We are looking for possible string solutions, with motion along the directions α, ψ, motivated by the fact that motion in these directions gives one BPS solution. Considering point-like strings and the simple ansatz[2]

$$\alpha = \omega_1\tau,\qquad \psi = \omega_3\tau, \tag{3.45}$$

we get only one non-trivial equation (3.2). This is solved by

$$y_\pm = 1 \pm \frac{\sqrt{(1-a)(2\omega_1 - \omega_3)(6\omega_1 + \omega_3)}}{\sqrt{3}(2\omega_1 - \omega_3)},\ \ \text{or} \tag{3.46}$$

$$\omega_1 = -\frac{\omega_3}{6}.$$

We do not examine solutions in the limits $a = 0,\ 1$ and in the rest of the analysis we will ignore solutions that are valid only there. The solution $\omega_1 = -\omega_3/6$ is BPS and is examined in a previous section. The other solution for y, is acceptable in a region which will be specified. Notice, that for

$$\omega_1 = -\frac{y+2}{6y}\omega_3\,, \tag{3.47}$$

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[2]Whenever we are writing an ansatz we suppose that all the parameters ω_i, m_i are non-zero.
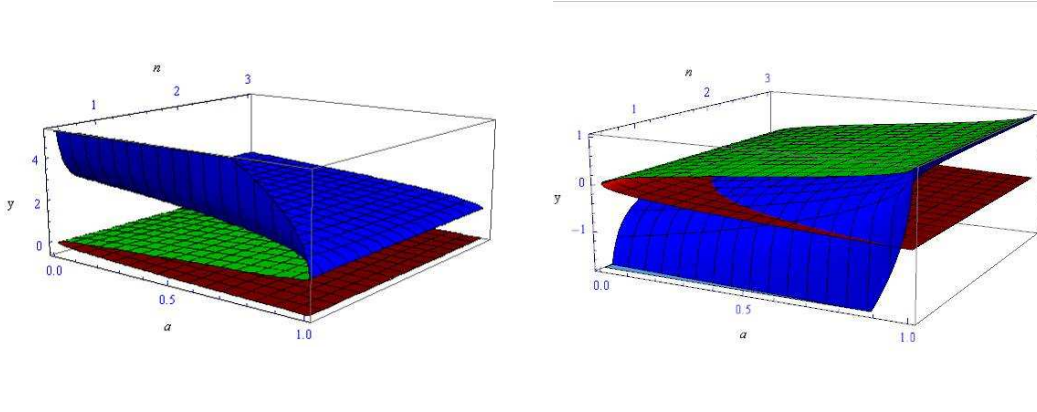
Figure 3. We plot $y_\pm$ versus n, a where $\omega_1 = n\omega_3$ with $n > 1/2$. In the first plot with y_+ we clearly see that there is no acceptable solution. In the second plot, the y_- give acceptable solutions.

which is the solution derived in (3.32), the solution (3.46) for y becomes BPS and lives on $y = y_{q\pm}$, but by considering the boundary conditions becomes static.

For the general case, first of all, one must find the values for which the square root is real and these are for

$$\omega_3 > 0, \ \ \omega_1 > \frac{\omega_3}{2} \ \text{ or } \ \omega_1 < -\frac{\omega_3}{6} \tag{3.48}$$

$$\omega_3 < 0, \ \ \omega_1 < \frac{\omega_3}{2} \ \text{ or } \ \omega_1 > -\frac{\omega_3}{6}. \tag{3.49}$$

Then we can see that indeed there exists solutions of y that are between y_{q-} and y_{q+}. Actually, the two solutions of (3.46) are equivalent depending on the sign of numbers ω_1, ω_3, i.e. if they are positive or negative. What one can say directly, is that the solution for y with r.h.s. equal to 'one minus a positive quantity', is the one that could be acceptable in some intervals. In order keep the presentation simpler we choose $\omega_1, \omega_3 > 0$. To give a visual picture of how the solutions behave we plot the surfaces for some random relation between ω_1, ω_3, say $\omega_1 = n\omega_3$ in figure 3.

We see that y_- is an acceptable solution for

$$a > \frac{4(7 + 18n)}{(1 + 6n)^3}, \tag{3.50}$$

which also implies $n > 1/2$ in order for a to be smaller than 1.

For the general case, the equation (3.8) gives

$$\kappa^2 = 4\omega_1^2 - \frac{(6\omega_1 + \omega_3)\sqrt{(1 - a)(2\omega_1 - \omega_3)(6\omega_1 + \omega_3)}}{3\sqrt{3}}, \tag{3.51}$$

where we have substituted the solution y.

The conserved charges for our solution, given from (3.10), (3.11), (3.12) are

$$J_\alpha = \sqrt{\lambda} \left(4\omega_1 + \frac{4(-3\omega_1 + \omega_3)}{3} \sqrt{\frac{(1-a)(6\omega_1 + \omega_3)}{3(2\omega_1 - \omega_3)}} \right) , \qquad (3.52)$$

$$J_\phi = -\sqrt{\lambda} \frac{\omega_3 c_\theta}{3} \sqrt{\frac{(1-a)(6\omega_1 + \omega_3)}{3(2\omega_1 - \omega_3)}} , \qquad (3.53)$$

$$J_\psi = \sqrt{\lambda} \frac{\omega_3}{3} \sqrt{\frac{(1-a)(6\omega_1 + \omega_3)}{3(2\omega_1 - \omega_3)}} , \qquad (3.54)$$

where (3.46) used. Notice the relation $J_\phi = -c_\theta J_\psi$. It seems that the energy depends on the momenta in a transcendental way. This is due to the complicated expression of y which depends on $\omega_{1,3}$, and enters in the momenta through the functions w, q, f. On the other hand, as we can see from (3.51), the energy is a monotonically increasing function with respect to a when ω_1, ω_3 are fixed. This behavior is similar to the one angle solution we found before, but it will be more interesting to have an energy-spin relation, which could be found by substituting in the second Virasoro constraint the $\omega_{1,3}$ with two conserved charges.

To finish, let us consider the example of $Y^{2,1}$. To find the corresponding value of a we can use (2.8) and find that

$$a = \frac{1}{4} \left(2 - \frac{\sqrt{13}}{8} \right) \simeq 0.387327 . \qquad (3.55)$$

Substituting in (3.50), we see that indeed we can get solutions for y that are inside the desirable interval when $\omega_1 > 0.8568 \omega_3$, or equivalently $n > 0.8568$. To simplify things even more we can choose $\theta = 0$, and solve for ω_1, ω_3 in terms of J_α, J_ψ to get the energy-spin relation. The energy-spin relation looks lengthy and complicated and seems to be completely transcendental, so we choose not to present it here.

Generalizing the ansatz by adding a σ dependence on the angle α and allowing the string to spin along this direction, we search for string solutions using

$$\alpha = \omega_1 \tau + m_1 \sigma, \qquad \psi = \omega_3 \tau. \qquad (3.56)$$

The non-trivial equations that needs to be solved in this case are (3.2) and (3.7). One acceptable solution is

$$\omega_1 = \frac{\omega_3}{6}, \qquad y = a, \qquad a = \frac{2\omega_3^2}{9m_1^2 + \omega_3^2} . \qquad (3.57)$$

By considering $a < 1$, and choosing ω_1, $\omega_3 > 0$, we get $|m_1| > \omega_3/3$. To satisfy the condition (2.5), we have to restrict m_1 further as

$$|m_1| > \frac{\omega_3}{\sqrt{3}} \Rightarrow a < \frac{1}{2} . \qquad (3.58)$$

Hence solutions that satisfy the above condition are acceptable as can be also seen clearly in figure 4. To calculate the energy we use (3.8) which gives
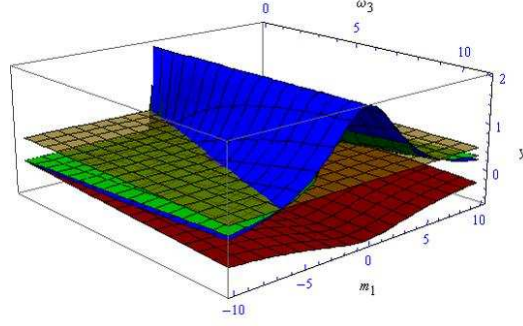
Figure 4. Plotting y, y_{q+}, y_{q-} versus ω_3, m_1. The transparent plane is at $a = 1/2$. When (3.58) satisfied, $a < 1/2$ and hence our solution lives in the area between the y_{q+}, y_{q-}. Also notice that for $a > 1$ there is no green surface, since the values of y_{q+} become complex, as expected.

$$\kappa^2 = \frac{(15m_1^2 - \omega_3^2)\omega_3^2}{3\left(9m_1^2 + \omega_3^2\right)}, \tag{3.59}$$

and the corresponding charges are given by

$$J_\alpha = 0, \qquad J_\phi = \sqrt{\lambda}\frac{-c_\theta\left(\omega_3^2 - 3m_1^2\right)}{3\left(9m_1^2 + \omega_3^2\right)}\omega_3, \qquad J_\psi = \sqrt{\lambda}\left(-\frac{1}{3} + \frac{4m_1^2}{9m_1^2 + \omega_3^2}\right)\omega_3. \tag{3.60}$$

Before we continue, we point out that one has to check if the function of a we get from (3.57) can give rational values to the expression $y_{q+} - y_{q-}$. This was not done in the previous cases since a was not dependent on ω or m, and hence it can take any appropriate value between zero and one, which would make the subtraction of the two roots of $q(y) = 0$ rational. In this case, one way to solve this problem is to equate the expression for a, given in (2.8), with our solution (3.57), and see if we can get rational solutions in the acceptable interval for ξ. One more reason to express ω_3 in terms of a is that is more preferable to choose a manifold $Y^{p,q}$, and from that to specify the allowed values for ω_i, m_i, instead of proceeding in the reverse direction. Using (2.8) and (3.57), we get for ω_3, m_3

$$\omega_3 = \pm\frac{3\sqrt{a}}{\sqrt{2-a}}m_1 = \pm\frac{3\sqrt{9 + \sqrt{9 - 3\xi^2}\left(-3 + 4\xi^2\right)}}{\sqrt{27 + (3 - 4\xi^2)\sqrt{9 - 3\xi^2}}}m_1 \equiv \pm Dm_1, \tag{3.61}$$

where D is defined from the above equation. We are going examine the solutions with the plus sign which imply $m_1 > 0$. For $0 < \xi < 3/2$, the multiplicative factor satisfies $0 < D < 3$, and by imposing the constraint (3.58), we are limited in the interval

$$\xi < \frac{\sqrt{3}}{2} \Leftrightarrow a < \frac{1}{2}, \tag{3.62}$$

which also means that $D < \sqrt{3}$. The corresponding conserved charges in terms of a, m_1 are

$$J_\alpha = 0, \qquad J_\phi = -\sqrt{\lambda}\frac{(1 - 2a)\sqrt{a}c_\theta}{3\sqrt{2-a}}m_1, \qquad J_\psi = \sqrt{\lambda}\frac{(1 - 2a)\sqrt{a}}{3\sqrt{2-a}}m_1, \tag{3.63}$$

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or using ω's

$$J_\phi = -\sqrt{\lambda}\frac{(1-2a)c_\theta}{9}\omega_3, \qquad J_\psi = \sqrt{\lambda}\frac{(1-2a)}{9}\omega_3 \ . \tag{3.64}$$

The sum of the momenta is

$$J_{\text{tot}} = \frac{\sqrt{\alpha}(2\alpha-1)(\cos\theta-1)}{3\sqrt{2-\alpha}}m_1 \ . \tag{3.65}$$

Finally, the relation between the charges is

$$J_\phi = -c_\theta J_\psi = -\frac{c_\theta}{1-c_\theta}J_{\text{tot}} \ . \tag{3.66}$$

Notice, that the zeroth momentum J_α can follow from a more general solution such as $y = a$ and $\omega_1 = \omega_3/6$.

To find the energy, we use (3.8) to get

$$\kappa^2 = \frac{a(5-4a)}{2-a}m_1^2 \ , \tag{3.67}$$

which can be written in terms of ω_3 by solving (3.61) for m_1. As before for a fixed ω_3, the energy is a monotonically increasing function with respect to a for $a < 1/2$. Using (3.63), we get the energy of our solution in terms of the momenta

$$E = \sqrt{\lambda}\sqrt{\frac{a(5-4a)}{2-a}m_1^2} = \frac{3\sqrt{5-4a}}{(1-2a)(1-c_\theta)}J_{\text{tot}} \ . \tag{3.68}$$

The angle θ is a constant, but in case we want to eliminate it, we can express c_θ in terms of the momenta from equation (3.66) and get

$$E = \frac{3\sqrt{5-4a}}{(1-2a)}J_\psi \ . \tag{3.69}$$

It is also interesting to insert the ξ parameters in the expressions we calculate using (3.61). Then the energy in terms of the momenta and the rational number ξ is

$$E = \frac{3\sqrt{3}}{(3-4\xi^2)(1-c_\theta)}\sqrt{\frac{27+2(3-4\xi^2)\sqrt{9-3\xi^2}}{3-\xi^2}}J_{\text{tot}} \ . \tag{3.70}$$

The equations (3.68) or (3.70) show that the energy is proportional to the momenta and depends on a. The exact dependence of the energy multiplied by $(1-c_\theta)$, in order to avoid the dependence on the third parameter θ and to be able to plot a surface, is presented in figure 5. We can see that the factor of proportionality between the energy and the momenta, is monotonically increasing with respect to a. However, the form of this factor does not seem to follow a specific pattern between the different string solutions. This occurs mainly because y depends on ω and m in a different way for each string solution. Also notice that the momenta are not taking continuous values but are quantized on a and m_1. Hence, the energy will not be a continuous function of the parameters, even if we plot it as continuous in order to show its behavior.

To finish the analysis for this case we mention that there are other solutions of (3.2) and (3.7), but these solutions are excluded since there is no a that satisfies the rationality constraint of $y_{q+} - y_{q-}$ and at the same time gives integer winding numbers.
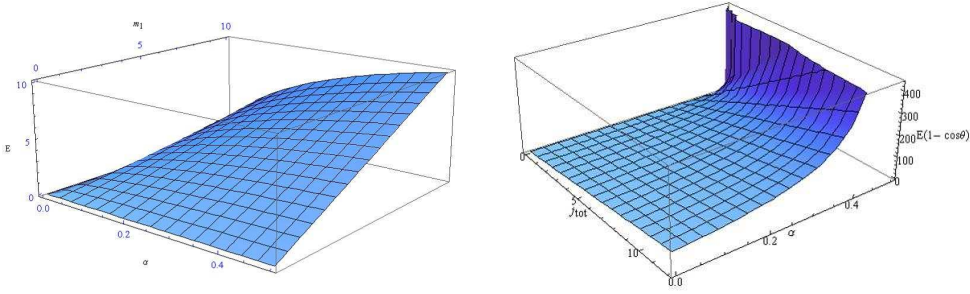
Figure 5. In the first plot is the energy versus m_1, a. In the second plot the expression $E \cdot (1 - c_\theta)$ is plotted versus J_{tot}, a, and one can notice the sharp rise of the energy as $a \to 1/2$. The plot range on energy is restricted intentionally to finite region in order to have a clear shape of the surface.

4 String solutions in $L^{p,q,r}$ background

4.1 Equations of motion and conserved quantities

In this section we construct solutions for strings moving on the $L^{p,q,r}$ manifold. The configuration is chosen to be similar with the analysis of the $Y^{p,q}$ manifold. Firstly, we choose the string to sit at a constant angle θ. In this manifold, the role of the previous y coordinate is played by the coordinate x, and so we restrict the string to be localized at a constant point x_0, which has to respect the constraint (2.18). For some configurations, the points θ and x that the string is sitting can be chosen arbitrary, but in most cases the equations of the system constrain at least one of them. As for the string dynamics, we are going to consider motion along some of the three U(1) directions and try to find the energy-spin relation and how the energy is related on the properties of the general manifold.

Furthermore, we are not going to analyze the string dynamics in the AdS_5 since these are identical to the maximally supersymmetric case. To simplify things we are localizing the string also at $\rho = 0$ on AdS_5 and expressing the global time through the world-sheet time by $t = \kappa\tau$. Thus, the Polyakov action in the conformal gauge is given by

$$
S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Bigg[-(-\dot{t}^2 + t'^2) + (-\dot{\xi}^2 + \xi'^2) + \frac{\rho^2}{4\Delta}(-\dot{x}^2 + x'^2) +
$$
$$
+ \frac{\rho^2}{h}(-\dot{\theta}^2 + \theta'^2) + \left(\frac{(\alpha - x)^2}{\alpha^2} s_\theta^2 + \frac{\Delta s_\theta^2 + h c_\theta^2 (\alpha - x)^2}{\rho^2 \alpha^2} \right) s_\theta^2 (-\dot{\phi}^2 + \phi'^2) +
$$
$$
+ \left(\frac{(\beta - x)^2}{\beta^2} c_\theta^2 + \frac{\Delta c_\theta^2 + h s_\theta^2 (\beta - x)^2}{\rho^2 \beta^2} \right) c_\theta^2 (-\dot{\psi}^2 + \psi'^2) +
$$
$$
+ 2 \left((\alpha - x)(\beta - x) + \frac{\Delta - h(\alpha - x)(\beta - x)}{\rho^2} \right) \frac{c_\theta^2 s_\theta^2}{\alpha\beta}(-\dot{\psi}\dot{\phi} + \psi'\phi') +
$$
$$
+ 2 \frac{\alpha - x}{\alpha} s_\theta^2 (-\dot{\xi}\dot{\phi} + \xi'\phi') + 2 \frac{\beta - x}{\beta} c_\theta^2 (-\dot{\xi}\dot{\psi} + \xi'\psi') \Bigg].
$$

Again, we do not write the dependence of the functions on their arguments, in order to sim-

plify the presentation. Also notice that α should not be confused with the same letter used in the case of $Y^{p,q}$ manifolds to name the angle. Before we start writing down the equations of motion, we define for convenience some new quantities. We identify the three expressions which are multiplied by $(-\dot\phi^2 + \phi'^2)$, $(-\dot\psi^2 + \psi'^2)$ and $(-\dot\psi\dot\phi + \psi'\phi')$, in the above action, which also correspond to metric elements, with the functions $d_1(x,\theta) \equiv g_{\phi\phi}$, $d_2(x,\theta) \equiv g_{\psi\psi}$ and $d_3(x,\theta) \equiv g_{\phi\psi}$ respectively. The partial derivatives of these functions are presented in appendix A, and will be used later.

The equations of motion for the θ, x are:

$$\partial_\theta d_1(-\dot\phi^2 + \phi'^2) + \partial_\theta d_2(-\dot\psi^2 + \psi'^2) + 2\partial_\theta d_3(-\dot\psi\dot\phi + \psi'\phi')$$
$$+2\frac{a-x}{a}s_{2\theta}(-\dot\xi\dot\phi + \xi'\phi') - 2\frac{\beta-x}{\beta}s_{2\theta}(-\dot\xi\dot\psi + \xi'\psi') = 0\,, \tag{4.1}$$

$$\partial_x d_1(-\dot\phi^2 + \phi'^2) + \partial_x d_2(-\dot\psi^2 + \psi'^2) + 2\partial_x d_3(-\dot\psi\dot\phi + \psi'\phi')$$
$$-\frac{2}{\alpha}s_\theta^2(-\dot\xi\dot\phi + \xi'\phi') - \frac{2}{\beta}c_\theta^2(-\dot\xi\dot\psi + \xi'\psi') = 0\,, \tag{4.2}$$

where we have used the fact that the string is localized at two fixed points θ_0, x_0. The equations of motion for the three U(1) directions ϕ, ψ, and ξ are

$$\partial_\kappa\left(\gamma^{\kappa\lambda}\left(d_1\partial_\lambda\phi + d_3\partial_\lambda\psi + 2\frac{\alpha-x}{\alpha}s_\theta^2\partial_\lambda\xi\right)\right) = 0\,, \tag{4.3}$$

$$\partial_\kappa\left(\gamma^{\kappa\lambda}\left(d_2\partial_\lambda\psi + d_3\partial_\lambda\phi + 2\frac{\beta-x}{\beta}c_\theta^2\partial_\lambda\xi\right)\right) = 0\,, \tag{4.4}$$

$$\partial_\kappa\left(\gamma^{\kappa\lambda}\left(2\frac{\alpha-x}{\alpha}s_\theta^2\partial_\lambda\phi + 2\frac{\beta-x}{\beta}c_\theta^2\partial_\lambda\psi + \partial_\lambda\xi\right)\right) = 0\,. \tag{4.5}$$

Note that they are trivially satisfied for the linear ansatz we choose.

Furthermore, the Virasoro constraints are given by

$$\dot\xi\xi' + d_1\dot\phi\phi' + d_2\dot\psi\psi' + d_3(\dot\psi\phi' + \psi'\dot\phi) + \frac{\alpha-x}{\alpha}s_\theta^2(\dot\xi\phi' + \xi'\dot\phi) \tag{4.6}$$

$$+\frac{\beta-x}{\beta}c_\theta^2(\dot\xi\psi' + \xi'\dot\psi) = 0\,,$$

$$\kappa^2 = \dot\xi^2 + \xi'^2 + d_1(\dot\phi^2 + \phi'^2) + d_2(\dot\psi^2 + \psi'^2) + 2d_3(\dot\psi\dot\phi + \psi'\phi') \tag{4.7}$$

$$+2\frac{\alpha-x}{\alpha}s_\theta^2(\dot\xi\dot\phi + \xi'\phi') + 2\frac{\beta-x}{\beta}c_\theta^2(\dot\xi\dot\psi + \xi'\psi').$$

Finally, the conserved charges associated to the three U(1) isometries are

$$J_\xi = \frac{\sqrt\lambda}{2\pi}\int_0^{2\pi} d\sigma\left(\dot\xi + \frac{\alpha-x}{\alpha}s_\theta^2\dot\phi + \frac{\beta-x}{\beta}c_\theta^2\dot\psi\right)\,, \tag{4.8}$$

$$J_\phi = \frac{\sqrt\lambda}{2\pi}\int_0^{2\pi} d\sigma\left(\frac{\alpha-x}{\alpha}s_\theta^2\dot\xi + d_1\dot\phi + d_3\dot\psi\right)\,, \tag{4.9}$$

$$J_\psi = \frac{\sqrt\lambda}{2\pi}\int_0^{2\pi} d\sigma\left(\frac{\beta-x}{\beta}c_\theta^2\dot\xi + d_3\dot\phi + d_2\dot\psi\right)\,, \tag{4.10}$$

where the classical energy is given by (3.9).

In the next section we use these equations to find some string solutions on $L^{p,q,r}$.

The ansatz for the string motion we consider have a linear dependence with τ and σ and are the following

$$\xi = \omega_1 \tau + m_1 \sigma, \qquad \phi = \omega_2 \tau + m_2 \sigma, \qquad \psi = \omega_3 \tau + m_3 \sigma \qquad (4.11)$$

$$\theta = \theta_0 \ \text{ and } \ x = x_0, \qquad (4.12)$$

where θ_0, x_0 are constants and we are also setting $\mu = 1$.

4.2 One angle solution

It is straight-forward to see that in the coordinate system we chose for the metric, there are solutions for point-like strings moving on the direction ξ, since the metric element $g_{\xi\xi}$ is constant. As we also commented in a previous section, a general property of the Virasoro constraints is that they do not allow any extended spinning string solutions, where the strings are moving only along one U(1) direction.

We start by considering the trivial case of $\xi = \omega_1 \tau$, where all the equations of motion and the first Virasoro constraint are satisfied trivially. The conserved momenta are

$$J_\xi = \sqrt{\lambda}\omega_1, \quad J_\phi = \sqrt{\lambda}\frac{\alpha - x}{\alpha}s_\theta^2\omega_1, \quad J_\psi = \sqrt{\lambda}\frac{\beta - x}{\beta}c_\theta^2\omega_1. \qquad (4.13)$$

The second Virasoro constraint gives the energy

$$E = \sqrt{\lambda}\sqrt{\omega_1^2} = |J_\xi| = \frac{2\alpha\beta}{4\alpha\beta - (\alpha + \beta)x - (\alpha - \beta)x \ c_{2\theta}}|J_{\text{tot}}|, \qquad (4.14)$$

which is presented in terms of the momenta. We see that the energy is proportional to the total spin and the factor of proportionality depends on α, β, and hence on the manifold $L^{p,q,r}$ considered.

Now consider a point-like string rotating along ϕ direction with $\phi = \omega_2 \tau$. The first two equations of motion give

$$\partial_\theta d_1(x, \theta) = 0, \qquad \partial_x d_1(x, \theta) = 0 \ . \qquad (4.15)$$

In the region that α, β are defined, the only real solution is when $\theta = 0$, which makes the $g_{\phi\phi}$ element zero. As a final possibility, consider the string moving according to $\psi = \omega_3 \tau$, which has to satisfy the equations

$$\partial_\theta d_2(x, \theta) = 0, \qquad \partial_x d_2(x, \theta) = 0 \ . \qquad (4.16)$$

These have a real solution only for $\theta = \pi/2$ which makes the metric element in the direction of rotation equal to zero.

For completeness we mention that the corresponding static string ansatze $m_i\sigma$, gives an acceptable solution only for a wrapping around the α direction, and in this case all the conserved momenta are zero.
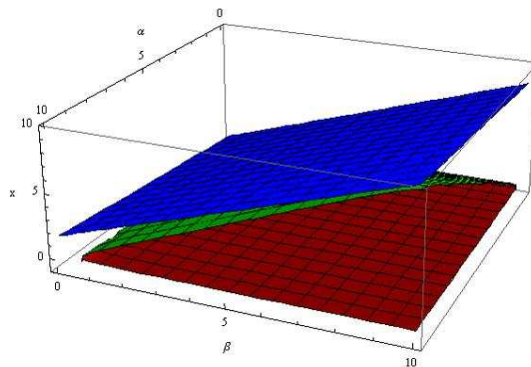
Figure 6. In the plot are presented the solutions of $\Delta(x)$, x_1 and x_2, together with x (4.19) versus the manifold parameters α, β. The solution x is plotted with blue and we see that there is no region such that x is between of x_1, x_2 where the manifold constraints are satisfied.

4.3 The two angle solutions

In this section we try to sketch a way of finding solutions of strings moving in two Killing vector directions simultaneously.

We choose to activate the angles ξ, ψ, and initially look for point-like string solutions

$$\xi = \omega_1 \tau, \qquad \psi = \omega_3 \tau. \tag{4.17}$$

The equations of motion reduce to

$$\partial_\theta d_2 \, \omega_3 - 2\frac{\beta - x}{\beta} s_{2\theta} \, \omega_1 = 0 \, , \qquad \partial_x d_2 \, \omega_3 - 2\frac{c_\theta^2}{\beta}\omega_1 = 0 \, , \tag{4.18}$$

where in order to check easier the inequality (2.18), and solve the above equations, it is convenient to give an appropriate value to angle θ. By choosing $\theta = \pi/4$ we get a solution

$$\alpha = \beta \pm \frac{3\omega_3}{\sqrt{2}\sqrt{-\beta\omega_3(2\omega_1 + \omega_3)}}, \qquad x = \frac{\alpha + 5\beta}{6} \, . \tag{4.19}$$

To make sure the solution is real we need to constrain ω_3 by

$$\omega_1 < 0 \quad \text{and} \quad 0 < \omega_3 < -2\omega_1 \quad \text{or} \quad \omega_1 > 0 \quad \text{and} \quad -2\omega_1 < \omega_3 < 0 \, . \tag{4.20}$$

Moreover, using the inequalities between x_2, α, and β we can easily see that $x_2 \leq (\alpha+5\beta)/6$, which means that x can only be equal to x_2. This seems to be a general feature for these solutions, since even for $\theta = \pi/3$, the situation is similar. But for a general angle θ, it is more difficult to solve the equations of motion and identify this behavior. However, in the cases mentioned above, the solution x is always strictly greater than x_2, and hence not acceptable (figure 6). There are also other solutions that could give acceptable answers, but need more extensive analysis.

To obtain an extended string configuration consider the ansatz

$$\alpha = \omega_1 \tau + m_1 \sigma, \quad \psi = \omega_3 \tau + m_3 \sigma, \tag{4.21}$$

which gives the following system of equations

$$\partial_\theta d_2(-\omega_3^2 + m_3^2) - 2\frac{\beta - x}{\beta} s_{2\theta}(-\omega_1 \omega_3 + m_1 m_3) = 0, \tag{4.22}$$

$$\partial_x d_2(-\omega_3^2 + m_3^2) - \frac{2}{\beta} c_\theta^2(-\omega_1 \omega_3 + m_1 m_3) = 0, \tag{4.23}$$

$$\omega_1 m_1 + d_2 \omega_3 m_3 + \frac{\beta - x}{\beta} c_\theta^2(\omega_1 m_3 + m_1 \omega_3) = 0. \tag{4.24}$$

For $\omega_1 = m_1$ and $\omega_3 = m_3$ reduces to a single equation

$$1 + d_2 \frac{\omega_3^2}{\omega_1^2} + 2\frac{\beta - x}{\beta} c_\theta^2 \frac{\omega_3}{\omega_1} = 0 \tag{4.25}$$

and the simpler solution is for $\theta = \pi/4$,

$$\omega_1 = -\frac{\omega_3}{2}, \qquad x = \frac{-2 + \alpha\beta^2 + \beta^3}{2\beta^2}. \tag{4.26}$$

In order this solution to satisfy the inequalities between α, β, and x, is essential for the following inequality $|\alpha - \beta| < 2/\beta^2$ to be satisfied. However, one can see that the solution is not acceptable because it is always greater than x_2. It is also interesting to examine the more complicated solutions, and see if they can satisfy (2.18).

Since it is complicated to check analytically if the solutions which involve a general angle θ satisfy the manifold constraints, we choose not to present them here, and maybe leave the problem for further investigation in the future. However, we already gave the basic setup with all the equations, found some pointlike solutions and also show the method to follow to find new string solutions moving in the U(1) directions of $L^{p,q,r}$ manifolds.

5 Conclusion

In this paper, we initially examine the string motion on $Y^{p,q}$ along the U(1) directions. To accept the solutions of the equations of motion and the Virasoro constraints, we must make sure that they also satisfy the Sasaki-Einstein constraints in a way we described above. Due to the presence of all these constraints, the number of acceptable string solutions is limited drastically. Then we show that when the energy is expressed in terms of the conserved momenta, the factors multiplied with them depend on the manifold, i.e. on the parameter a, where they are monotonic functions with respect to it. Hence, the extrema of these functions occur for maximum or minimum values of a. Except this behavior, it seems that the dispersion relations depend on the parameter a in a transcendental way.

Furthermore, by looking at massless geodesics, we find that there is a unique BPS solution, which was already known. What we see here is that the string coordinates must depend linearly with time. Moreover, for this solution, the string can sit anywhere in the

allowed y interval satisfying (2.5). An other point-like string solution was found which lives on the two supersymmetric three-submanifolds $S^3/\mathcal{Z}_{p+q}, S^3/\mathcal{Z}_{p+q}$, obtained by the initial manifold for $y = y_{q\pm}$. However, by considering the boundary conditions, the solution becomes static, and we argue that there are no other point-like BPS solutions in the analysis presented above.

One can work similarly in the cohomogeneity two manifolds $L^{p,q,r}$, finding some point-like and classical string solutions. Again, we expect that when the energy is expressed in terms of the conserved momenta, it does not take the same form uniformly over the family of manifolds $L^{p,q,r}$, and basically the discussion we presented for the $Y^{p,q}$ manifold remain similar with the $L^{p,q,r}$ manifold. However, in this case, due to the large number of parameters, it is more difficult to check analytically in full generality whether or not the solutions satisfy the manifold constraints. One can certainly try to find some more solutions for these manifolds. Furthermore, the solutions of $Y^{p,q}$ found in the previous section should correspond to some solutions in $L^{p,q,r}$ manifolds. To establish a direct way to correspond these solutions one possibly needs to make some suitable coordinate transformations.

An interesting extension of this work could be to consider strings having σ dependence on one of the y, θ angles. By activating simultaneously one more U(1) angle for the string's motion, the analysis should not be difficult. However, simultaneous string motion on more directions, could lead to systems of differential equations that might be difficult to solve. Also, worth looking at, is the effect of the β deformations on the string solutions on these manifolds. One can initially work with the point-like BPS solutions presented above, and try to derive the β deformed 'BPS condition'. It can be checked then if the BPS massless geodesics found above, remain undeformed. It must be possible to support these results accordingly in the dual beta deformed theory. Another interesting topic is to examine the string motion simultaneously in several U(1) directions, and to analyze the energy-spin relations. Finally, it would be very good to identify the solutions found in this paper with the corresponding operators in field theory. Some of the above issues will be examined in a forthcoming publication [37].

A   Some formulas of Sasaki-Einstein manifolds

The three roots of cubic satisfy

$$y_{q+} + y_{q-} + y_3 = 3/2, \quad y_{q+}y_{q-} + y_{q+}y_3 + y_{q-}y_3 = 0, \quad 2y_{q+}y_{q-}y_3 = -a \qquad \text{(A.1)}$$

and also can be expressed in terms of p, q

$$y_{q\pm} = \frac{1}{4p}\left(2p \pm 3q - \sqrt{4p^2 - 3q^2}\right), \quad y_3 = \frac{1}{4p}\left(2p + 2\sqrt{4p^2 - 3q^2}\right) \qquad \text{(A.2)}$$

and the period of a (2.10), can be rewritten in a more compact form

$$l = -\frac{q}{4p^2 y_{q+}y_{q-}}, \qquad \text{(A.3)}$$

which is always positive since y_{q-} is negative. The volume $Y^{p,q}$ is given by

$$Vol(Y^{p,q}) = \frac{q\left(2p + \sqrt{4p^2 - 3q^2}\right)l\pi^3}{3p^2} \qquad \text{(A.4)}$$

and is bounded by

$$Vol(T^{1,1}/Z_p) > Vol(Y^{p,q}) > Vol(S^5/Z_2 \times Z_p). \tag{A.5}$$

B Definition of functions used in $L^{p,q,r}$ string solutions

In order to make the presentation shorter we define the following functions as

$$d_1(x,\theta) \equiv \left(\frac{(\alpha-x)^2}{\alpha^2} s_\theta^2 + \frac{\Delta s_\theta^2 + h c_\theta^2 (\alpha-x)^2}{\rho^2 \alpha^2} \right) s_\theta^2 \,, \tag{B.1}$$

$$d_2(x,\theta) \equiv \left(\frac{(\beta-x)^2}{\beta^2} c_\theta^2 + \frac{\Delta c_\theta^2 + h s_\theta^2 (\beta-x)^2}{\rho^2 \beta^2} \right) c_\theta^2 \,, \tag{B.2}$$

$$d_3(x,\theta) \equiv \left((\alpha-x)(\beta-x) + \frac{\Delta - h(\alpha-x)(\beta-x)}{\rho^2} \right) \frac{c_\theta^2 s_\theta^2}{\alpha\beta} \,, \tag{B.3}$$

which have the following partial derivatives. With respect to θ:

$$\partial_\theta d_1(x,\theta) = \frac{s_{2\theta}}{\alpha^2(\alpha-\beta)} \left(\mu + \alpha(\alpha-\beta)(\alpha-x) - \frac{4\mu(\alpha-x)^2}{(\alpha+\beta-2x+(\alpha-\beta)c_{2\theta})^2} \right) \,,$$

$$\partial_\theta d_2(x,\theta) = \frac{s_{2\theta}}{\beta^2(\alpha-\beta)} \left(\mu + \beta(\beta-\alpha)(\beta-x) - \frac{4\mu(\beta-x)^2}{(\alpha+\beta-2x+(\alpha-\beta)c_{2\theta})^2} \right) \,,$$

$$\partial_\theta d_3(x,\theta) = \frac{\mu s_{2\theta} \left((x-\alpha c_\theta^2) c_\theta^2 - (x-\beta s_\theta^2) s_\theta^2 \right)}{\alpha\beta \left(-x + \alpha c_\theta^2 + \beta s_\theta^2 \right)^2}$$

and with respect to x:

$$\partial_x d_1(x,\theta) = -\frac{s_\theta^2}{8\alpha^2 \left(-x + \alpha c_\theta^2 + b s_\theta^2 \right)^2} \Big(4\mu + \alpha \left(3\alpha^2 + 2\alpha\beta + 3\beta^2 - 8(\alpha+\beta)x + 8x^2 \right) +$$
$$+ 4(-\mu + \alpha(\alpha-\beta)(\alpha+\beta-2x))c_{2\theta} + \alpha(\alpha-\beta)^2 c_{4\theta} \Big) \,, \tag{B.4}$$

$$\partial_x d_2(x,\theta) = -\frac{c_\theta^2}{8\beta^2 \left(-x + \alpha c_\theta^2 + \beta s_\theta^2 \right)^2} \Big(4\mu + \beta \left(3\alpha^2 + 2\alpha\beta + 3\beta^2 - 8(\alpha+\beta)x + 8x^2 \right) +$$
$$+ 4(\mu + (\alpha-\beta)\beta(\alpha+\beta-2x))c_{2\theta} + (\alpha-\beta)^2 \beta c_{4\theta} \Big) \,, \tag{B.5}$$

$$\partial_x d_3(x,\theta) = -\frac{\mu s_{2\theta}^2}{4\alpha\beta \left(-x + \alpha c_\theta^2 + \beta s_\theta^2 \right)^2} \,.$$

We use the above expressions to write the equations of motion and the Virasoro constraints in a more compact form.

Acknowledgments

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